



Haeffliger's codimension-one singular foliations, open books and twisted open books in dimension 3

Francois Laudenbach, Gael Gael Meigniez

► To cite this version:

Francois Laudenbach, Gael Gael Meigniez. Haeffliger's codimension-one singular foliations, open books and twisted open books in dimension 3. Boletim da Sociedade Brasileira de Matemática / Bulletin of the Brazilian Mathematical Society, 2012, 43 (3), pp.347-373. hal-00614589

HAL Id: hal-00614589

<https://hal.science/hal-00614589>

Submitted on 12 Aug 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HAEFLIGER'S CODIMENSION-ONE SINGULAR FOLIATIONS, OPEN BOOKS AND TWISTED OPEN BOOKS IN DIMENSION 3

FRANÇOIS LAUDENBACH AND GAËL MEIGNIEZ

ABSTRACT. We consider singular foliations of codimension one on 3-manifolds, in the sense defined by André Haefliger as being Γ_1 -structures. We prove that under the obvious linear embedding condition, they are Γ_1 -homotopic to a regular foliation carried by an open book or a twisted open book. The latter concept is introduced for this aim. Our result holds true in every regularity C^r , $r \geq 1$. In particular, in dimension 3, this gives a very simple proof of Thurston's 1976 regularization theorem without using Mather's homology equivalence.

1. INTRODUCTION

In this paper we prove a regularization theorem for singular foliations of codimension one on closed 3-manifolds. They will even be homotopic to some specific regular models.

Theorem 1.1. *Let M be a closed 3-manifold equipped with a Haefliger structure ξ of codimension one and differentiability class C^r , $1 \leq r \leq \infty$. Assume that the bundle normal to ξ embeds into the bundle tangent to M . Then ξ is C^r -homotopic to a regular Haefliger structure (that is, a foliation) carried by an open book decomposition of M , or by a twisted open book decomposition of M .*

Let us recall from [7] that, on a manifold M , a Haefliger structure ξ of codimension one, or a Γ_1 -structure, consists of:

- a line bundle $\nu = (E(\nu) \rightarrow M)$, that is, a real vector bundle of rank 1, called the bundle normal to ξ ;
- in the total space $E(\nu)$, a germ, still noted ξ , of codimension-one foliation along the zero section i of ν , transverse to the fibers.

The Γ_1 -structure ξ is said to be *regular* when the foliation ξ is also transverse to the zero section i . Then the trace $\xi \cap i(M)$ is a genuine foliation of M . If not, this trace is a singular foliation.

A *homotopy* of ξ is defined as a Γ_1 -structure on $M \times [0, 1]$ inducing ξ on $M \times \{0\}$. A *regularization* of ξ is a homotopy to a regular Γ_1 -structure. It does not exist in general. An obvious necessary condition is that ν must embed into the tangent bundle τM . When ν is trivial and $\dim M = 3$ this condition is fulfilled.

In what follows, the manifold M will be C^∞ and ξ will be a Γ_1^r -structure ($1 \leq r \leq \infty$), meaning that ξ viewed as a foliation is tangentially C^∞ and transversely C^r , that is, the foliation charts are C^r in the direction transverse to the leaves.

2000 *Mathematics Subject Classification.* 57R30.

Key words and phrases. Foliations, Haefliger's Γ -structures, open book.

FL is supported by ANR Floer Power.

Of course, “carried by a (twisted) open book decomposition” needs explanation. But we first comment on the regularization aspect only.

Under this aspect, our result is a particular case of a general regularization theorem due to W. Thurston (see [18]). Thurston’s proof was based on a deep result due to J. Mather [12, 13], namely, the homology isomorphism between on the one hand the classifying space of the group $\text{Diff}_c(\mathbb{R})$ endowed with the discrete topology, and on the other hand the loop space $\Omega B(\Gamma_1)_+$.

We present here a proof of the above regularization theorem which does not depend on the latter result. After the present research was achieved, a regularization theorem in all dimensions, still avoiding any difficult result, was provided in [15], without the open book models that we get here in dimension 3.

Our proof provides model foliations in all homotopy classes of Γ_1 -structure. In this introduction, to make short, we explain the model in the co-orientable case: that is, ν is trivial.

In this case, the models are based on the notion of *open book decomposition*. Recall that such a structure on M consists of a link B in M , called the *binding*, and a fibration $p : M \setminus B \rightarrow S^1$ such that, for every $\theta \in S^1$, the fiber $p^{-1}(\theta)$ is the interior of an embedded surface, called the *page* P_θ , whose boundary is the binding. The existence of an open book decomposition could have been proved by J. Alexander when M is orientable, as a consequence of [1] (every orientable closed 3-manifold is a branched cover of the 3-sphere) and [2] (every link can be braided); but the concept was introduced by H. Winkelnkemper in 1973 [21]. Henceforth, we refer to the more flexible construction by E. Giroux, which includes the case when M is non-orientable (see section 3).

It is well-known that every open book \mathcal{B} gives rise to a foliation $\mathcal{F}_\mathcal{B}$, as follows. The pages endow B with a normal framing. So, a tubular neighborhood $N(B)$ of B is trivialized: $N(B) \cong B \times D^2$. Out of $N(B)$ the leaves of $\mathcal{F}_\mathcal{B}$ are the pages modified by spiraling around $N(B)$; some neighborhood of $\partial N(B)$ in $N(B)$ is a union of compact leaves; and the rest of $N(B)$ is foliated by a Reeb component.

This foliation of $N(B)$ will be called a *thick Reeb component* and it is introduced for technical reasons in the homotopy argument of section 5. This technical point could be avoided by using a theorem of F. Sergeraert [16] on Reeb components. We call such a foliation an *open book foliation*.

The latter foliation can be modified by inserting a so-called *suspension foliation*. Precisely, let K be a compact subsurface in some leaf of $\mathcal{F}_\mathcal{B}$ out of $N(B)$, and let $K \times [-\varepsilon, +\varepsilon]$ be a *foliated thickening* of it: each $K \times \{t\}$ is contained in a leaf of $\mathcal{F}_\mathcal{B}$. Let $\varphi : \pi_1(K) \rightarrow \text{Diff}_c(-\varepsilon, +\varepsilon)$ be some representation into the group of compactly supported diffeomorphisms; φ is assumed to be trivial on the peripheral elements (necessarily ∂K is non-empty). It allows us to construct a *suspension foliation* \mathcal{F}_φ on $K \times [-\varepsilon, +\varepsilon]$, whose leaves are transverse to the vertical segments $\{x\} \times [-\varepsilon, +\varepsilon]$ and whose holonomy is φ . The modification consists of removing $\mathcal{F}_\mathcal{B}$ from the interior of $K \times [-\varepsilon, +\varepsilon]$ and replacing it by \mathcal{F}_φ . The new foliation, denoted $\mathcal{F}_{\mathcal{B},\varphi}$, is an *open book foliation modified by suspension*. It is also said to be *carried* by the open book decomposition \mathcal{B} .

When working with a Γ_1 -structure ξ whose normal bundle is twisted, it is necessary to introduce the notion of *twisted open book*, which is inspired by work of E. Giroux and will be

explained in section 3. It is worth noticing that the existence of twisted open book decompositions on M with a given *normal bundle* ν follows from our regularization theorem applied to the so-called *twisted trivial* Γ_1 -structure with normal bundle ν (see definition 2.2 and theorem 1.2).

The paper is organized as follows (see the text for definitions). In section 2, the given Γ_1^r -structure ξ is modified by a C^0 -small homotopy to make it Morse (that is, with Morse singularities), and to give it a *pseudo-gradient* X whose dynamics is trivial. We deduce a decomposition of M of the form $M = N(\Sigma) \cup H$, where $N(\Sigma)$ is a tubular neighborhood of an embedded closed surface Σ which is transverse to X , and where H is a handlebody with one or two connected components. Moreover, Σ is Poincaré-dual to $w_1(\nu)$, the first Stiefel-Whitney class of ν .

In section 3, we build an open book \mathcal{B} (possibly twisted) which is somewhat adapted to Σ : the surface Σ minus a disk is contained in a page, and the bundle normal to \mathcal{B} is isomorphic to ν . In passing, we obtain the following theorem:

Theorem 1.2. *Let ν be a line bundle over M which embeds into the tangent space τM . Then there exists a twisted (or non-twisted) open book decomposition \mathcal{B} of M whose normal bundle is isomorphic to ν . It is twisted whenever ν is so.*

In section 4, we apply to ξ a new homotopy to put it into *plateau form*, meaning that it is *trivial* in H , and in the interior of $N(\Sigma)$, it is transverse to X .

In section 5 a final homotopy puts ξ into the desired form $\mathcal{F}_{\mathcal{B},\varphi}$ carried by \mathcal{B} .

Many arguments of this paper become significantly simpler when ν is trivial. The reader who is mainly interested in this case may refer to [11]. Also, the reader interested in prescribing the homotopy class of the plane field tangent to the resulting foliation is referred to the same preprint.

We are very grateful to Vincent Colin, Étienne Ghys and Emmanuel Giroux for their comments, suggestions and explanations.

2. MORSIFICATION OF THE SINGULARITIES AND DYNAMICS OF A PSEUDO-GRADIENT

For proving theorem 1.1, the setting is a closed 3-manifold M endowed with a Γ_1^r -structure ξ , with $r \geq 1$. Nevertheless, the next proposition 2.1 holds true in any dimension.

Denote by ν the normal bundle to ξ , by $E(\nu)$ its total space, and by $i : M \rightarrow E(\nu)$ the zero section. Regard ξ as a foliation on some neighborhood of $i(M)$ in $E(\nu)$.

A point $x \in M$ is said to be a *singularity* of ξ when the zero section i is not transverse to ξ at x . One says that ξ is *Morse* when at each singularity the contact of i with ξ is quadratic non-degenerate in some foliated chart where i is C^2 . In that case, the trace $\xi \cap i(M)$ is a singular foliation which is locally defined by a Morse function.

For every section s of ν of class C^r which is C^0 -close to i , the pullback $s^*\xi$ is a Γ_1^r -structure on M homotopic to ξ . Such a homotopy of ξ is said to be a C^0 -small homotopy.

Proposition 2.1. *After a C^0 -small homotopy, ξ is Morse.*

Proof. Of course, in case $r \geq 2$ this proposition is obvious: any C^r -generic choice of s in a small enough C^0 -neighborhood of i works. In case $r = 1$ one has the following method which works in any dimension of M .

At the beginning the zero section is covered by finitely many boxes bi-foliated with respect to ξ and to the \mathbb{R} -fibers. These boxes form a C^r atlas. In each chart, the zero section becomes the graph of a real function of class C^r . We choose a triangulation Tr of $M = i(M)$ so fine that each simplex lies in such a box. Then, for each simplex σ in Tr we choose a bi-foliated neighborhood V_σ such that when τ is a face of σ we have $V_\tau \subset V_\sigma$. We need the two following lemmas.

Lemma 1. *Let D^k be the standard compact k -disk and let U be an open neighborhood in D^k of the boundary ∂D^k . Let $f : U \rightarrow \mathbb{R}$ be a function of class C^1 without critical points. Let V and W be compact collar neighborhoods of ∂D^k in D^k with:*

$$W \subset \text{int } V \subset V \subset U$$

Then there exists a real function $g : D^k \rightarrow \mathbb{R}$ such that:

- g coincides with f on W and has no critical point in V ;
- g is C^∞ on the complement of V ;
- g is a Morse function (which makes sense since g has no critical points in V).

Proof. Let X be a continuous vector field on U such that $X \cdot f > 0$. Let α be some smooth bump function on D^k which equals 1 on W and has support in V . Let $c > 0$ be less than the minimum of $X \cdot f$ on V . Let $m > 0$ be the maximum of $|X \cdot \alpha|$ on V . We choose a C^∞ function $h : D^k \rightarrow \mathbb{R}$ whose restriction to V is very close to f in the C^1 topology. More precisely, we require that the following inequalities hold on V :

- $X \cdot h > c$
- $m|f - h| < \frac{c}{2}$.

The function h which has no critical points on V may be chosen to be a Morse function on D^k . Then $g := \alpha f + (1 - \alpha)h$ has the wanted properties. Indeed, at every point of V ,

$$X \cdot g = \alpha X \cdot f + (1 - \alpha)X \cdot h + (X \cdot \alpha)(f - h)$$

which is larger than $c/2$. □

Lemma 2. *Let D^k be a compact k -disk embedded in M . Let \tilde{f} be a real C^1 function defined on some neighborhood U of ∂D^k in M . The functions \tilde{f} and $\tilde{f}|_{U \cap D^k}$ are assumed to have no critical points. Let $g : D^k \rightarrow \mathbb{R}$ be a Morse function which extends $\tilde{f}|_{U \cap D^k}$. Then, there are a neighborhood V of D^k in M and a function $\tilde{g} : V \rightarrow \mathbb{R}$ with the following properties:*

- $\tilde{g}|_{D^k} = g$;
- \tilde{g} coincides with \tilde{f} in a neighborhood of ∂D^k ;
- \tilde{g} is a Morse function and its critical points are those of g .

Proof. Near each critical point of g , the extension \tilde{g} is constructed by adding a non-degenerate quadratic form in the coordinates of a transversal to D^k . Hence, the last condition is satisfied. Away from the critical points we have to solve the following extension problem: we are given a submanifold N and a function $g : N \rightarrow \mathbb{R}$ without critical points. We have to extend g to a

neighborhood of N , the germ of the extension being already given near ∂N . It is easy by using the projection $p : V \rightarrow N$ of some tubular neighborhood of N in M , well chosen near ∂N , and setting $\tilde{g} = g \circ p$. \square

We now return to the proof of proposition 2.1. It is done by induction on the skeleta of Tr . Assume we already have a section s with the following property: for each simplex τ of dimension less than k , the restriction $(s^*\xi)|_\tau$ is Morse, and there is a neighborhood U of the $(k-1)$ -skeleton such that $s^*\xi$ is non-singular on $U \setminus Tr^{[k-1]}$. If σ is a k -simplex, $s|_\sigma$ viewed in the foliated chart V_σ is the graph of a function f to which it is allowable to apply successively the two preceding lemmas. It is clear from the construction that the new section is C^0 -close to the initial zero section i . \square

For example, the Morsification applies to the *twisted trivial* Γ_1 -structure in the following sense.

Definition 2.2. *Let ν be a line bundle over M , whose first Stiefel-Whitney class is viewed as a representation $w_1(\nu) : \pi_1(M) \rightarrow \mathbb{Z}/2 \subset GL_1(\mathbb{R})$. The twisted trivial Γ_1 -structure ξ_0 with normal bundle ν is the foliation of $E(\nu)$ which is the suspension of $w_1(\nu)$.*

Then, all points of M are singular. In other words, the zero section is a leaf. For every loop γ in this leaf, its holonomy is non-trivial if and only if ν is a twisted bundle over γ .

In what follows, ξ is assumed to be Morse. We are interested in the dynamics of a so-called *pseudo-gradient* of ξ , defined as follows. A *twisted* vector field X on M is a C^∞ section X of $Hom(\nu, \tau M)$. Then, the sign of $X \cdot \xi$ is well-defined at each point of $M = i(M)$ where X is transverse to ξ .

Definition 2.3. *A pseudo-gradient for ξ is a twisted vector field X on M such that the Lyapunov inequality $X \cdot \xi < 0$ holds everywhere but the singularities.*

As ν is possibly non-orientable, one cannot distinguish the index of a singularity from its co-index; but one can distinguish saddle points from center points thanks to their phase portraits. >From each saddle point s start two *separatrices*, that is, isolated orbits of X that begin at s .

Proposition 2.4. *After some C^0 -small homotopy, ξ is still Morse and admits a pseudo-gradient X whose dynamics has no recurrence (that is, every orbit has a finite length) and no separatrix joins two saddle points.*

Proof. The existence of a smooth pseudo-gradient is easy to prove even if ξ is C^1 only. Indeed, near the singularities, there are smooth charts of the singular foliation and the usual negative gradient in Morse coordinates is convenient. Away from them, the Lyapunov inequality allows one to approximate a C^0 gradient by a smooth pseudo-gradient. Let X_0 be such a pseudo-gradient; its dynamics is not controlled.

Finitely many mutually disjoint open 2-disks, d_1, \dots, d_N , are chosen in regular leaves of the trace $\xi \cap i(M)$ such that every orbit of X_0 crosses at least one shrunk disk $\frac{1}{2}d_k$, for some $k \in 1, \dots, N$. Following Wilson's idea [20], the zero section i of ξ and X_0 are changed in a neighborhood $D^2 \times [-1, +1]$ of each disk into a *plug* such that every orbit of the modified pseudo-gradient X is trapped by one of the plugs. Here are a few more details.

In these neighborhoods $D^2 \times [-1, +1]$, whose last coordinate is t , the foliation $\xi \cap i(M)$ is defined by the function t . By a C^0 -small homotopy supported in $D^2 \times [-1, 0]$, we just change the zero section i to a section s such that the singular foliation $\xi \cap s(M)$ is made of a cancelling pair of singularities, center–saddle. More precisely, it is defined by a Morse function f which coincides with t near the boundary and which has a pair of critical points of indices $(2, 3)$. In these cylinders, the new pseudo-gradient X is the gradient of f for the flat metric. By using $D^2 \times [0, 1]$ in a convenient way, it is easy to make the plug have the mirror symmetry with respect to $D^2 \times \{0\}$: any orbit of X entering the cylinder at $p \times \{-1\}$ is trapped if $|p| \leq \frac{3}{4}$ or gets out the cylinder at $p \times \{+1\}$. In the cylinder $D^2 \times [0, 1]$, the singular foliation $\xi \cap s(M)$ is defined by a Morse function with a pair of critical points of indices $(0, 1)$.

Thanks to the mirror symmetry, very much like in Wilson’s paper, the new twisted vector field X has a *finiteness property*:

Each orbit of X has a finite length, and connects two singularities.

Let us give a proof. We consider any half orbit λ of X , starting at some point x outside the plugs. We have to prove that the ω -limit set $\omega(\lambda)$ is a singularity of X . Let λ_0 be the half orbit of X_0 through x , with the same germ as λ at x . If λ is trapped by some plug, that is, if it ends at some singularity in this plug, then we are done. So, we may assume that λ is not trapped by any plug. Due to the mirror symmetry, as recalled before, whenever λ enters a plug at $p \times \{\mp 1\}$, $p \in D^2$, it gets out at $p \times \{\pm 1\}$. In particular, outside the plugs, we have $\lambda = \lambda_0$. It remains to show that $\omega(\lambda_0)$ is a singularity of X_0 . Indeed, if not, $\omega(\lambda_0)$ would contain a whole orbit of X_0 , thus would intersect some disk $\frac{1}{2}d_k$. Thus λ would be trapped by the corresponding plug, a contradiction.

Unfortunately, the mirror symmetry in the plugs implies that X has some separatrices connecting two saddle points. Nevertheless, one can destroy these connecting separatrices by a perturbation of X . We claim that the finiteness property is preserved when the perturbation is small enough.

By contradiction, assume that X is the C^0 -limit of some sequence (X_n) of twisted vector fields such that each X_n has an orbit segment λ_n of length n . Then the Hausdorff accumulation set of (λ_n) in M for $n \rightarrow +\infty$ would contain either a half orbit of X of infinite length, which does not exist by the finiteness property; or a half infinite (or periodic) broken orbit Λ of X : that is, Λ is an infinite (or periodic) sequence of orbits connecting successive saddle singularities. Then, if Λ enters a plug at $p \times \{\mp 1\}$ it must get out at $p \times \{\pm 1\}$. Just as in the above proof of the finiteness property for X , we conclude that there is some half infinite (or periodic) broken orbit Λ_0 of X_0 such that $\Lambda = \Lambda_0$ outside the plugs.

Let λ_0 be one of the orbits of X_0 which are contained in Λ_0 . Then λ_0 meets some disk $\frac{1}{2}d_k$. Thus, Λ is trapped by the corresponding plug and ends at the corresponding center singularity. Since Λ is contained in the accumulation set of (λ_n) , infinitely many λ_n ’s are trapped in the same way as λ_0 and their lengths are bounded, a contradiction. □

>From the pseudo-gradient X of ξ given by proposition 2.4 we deduce the following topological objects. Let G be the topological closure in M of the separatrices of all saddle points.

It is a graph whose vertices (resp. midpoints of the edges) are the center (resp. saddle) singularities of ξ . Thus G admits an arbitrarily small tubular neighborhood H whose boundary is transverse to X . Let \hat{M} be the complement of $\text{int } H$ in M . Since each orbit of X from a point of $\partial H = \partial \hat{M}$ has a finite length, it must return to the boundary. Therefore, \hat{M} is fibered over a surface Σ , $\rho : \hat{M} \rightarrow \Sigma$, the fibers being intervals ($\cong [-1, 1]$) tangent to X . By taking a section we think of Σ as a closed surface embedded in $M \setminus H$ and \hat{M} becomes a tubular neighborhood $N(\Sigma)$ of Σ in M . By construction of X , the normal bundle to Σ in M is $\nu|\Sigma$.

Proposition 2.5.

- 1) *The line bundle $\nu|G$ is orientable and, for a suitable choice of the orientation, X enters H along ∂H .*
- 2) *When ν is orientable, G has two connected components and Σ is two-sided.*
- 3) *When ν is non-orientable, G is connected and Σ is one-sided.*
- 4) *If ν embeds into the tangent bundle τM , the Euler characteristic of H is even.*

Proof. 1) We orient each separatrix from its saddle end point to its center end point. Since the separatrices are transverse to ξ , this is an orientation of $\nu|G$ over the complement of the singularities. It is easily checked that this orientation extends over the singularities. Thus X becomes a usual vector field near G transverse to ∂H .

2) When ν is a trivial bundle, by the isomorphism above-mentioned, $\nu(\Sigma, M)$ is trivial and Σ is two-sided. Thus, $\partial \hat{M}$ has two connected components and G also does.

3) Assume Σ be two-sided. Then $\hat{M} \cong [-1, 1] \times \Sigma$ and G has two connected components since each connected component of H has a connected boundary. In that case, ν is orientable since there are no arcs in H from $\Sigma \times \{+1\}$ to $\Sigma \times \{-1\}$.

4) By assumption, the pseudo-gradient X is homotopic to a non-vanishing one. Then the number of zeroes of X is even. \square

As a conclusion of this section, we shall be able to continue the proof of theorem 1.1 with a ξ having a pseudo-gradient X for which the following statement holds true.

Corollary 2.6. *The above decomposition $M = N(\Sigma) \cup H$ has the following properties:*

- 1) *The handlebody H has two or one components depending on that ν is trivial or not.*
- 2) *The restricted bundle $\nu|H$ is trivial; equivalently, Σ is Poincaré dual to $w_1(\nu)$. For a convenient orientation, X enters H along ∂H .*
- 3) *The fibres of $N(\Sigma) \rightarrow \Sigma$ are contained in orbits of the pseudo-gradient X .*

3. OPEN BOOKS, TWISTED OPEN BOOKS

E. Giroux explained to us a Morse theoretical approach to open book decomposition, which is based on [4] and is recalled below, up to a change of terminology. An easy adaptation allows us to handle similarly the twisted open book decompositions.

Definition 3.1. *Let W be a compact 3-manifold and $f : W \rightarrow \mathbb{R}$ be a Morse function which is constant and maximal on ∂W . A compact surface S properly embedded in W (that is, $\partial S \subset \partial W$) will be called a Giroux surface for f when $f|S$ is a Morse function having the same critical points and the same isolated local extrema as f .*

The saddle points of $f|_S$ may have index 1 or 2 for f . The property for f of having a Giroux surface is kept when deforming f among the Morse functions, even when crossing the critical values; for instance, through a deformation ending at a *self-indexing* Morse function (the value of a critical point is its Morse index in W). This definition is made for $W \subset M$; of course, we are mainly interested in the case when W is closed ($W = M$).

Theorem III.2.7 in Giroux's article [4] states the following with different words:

Theorem 3.2. *Let M be a closed 3-manifold (orientable or not). There exist a self-indexing Morse function $f : M \rightarrow \mathbb{R}$ and a surface S which is a 2-sided Giroux surface in M for f .*

In that case, S is separating (see below) and this data immediatly gives rise to an open book decomposition \mathcal{B} in the sense that is recalled in the introduction. Indeed, let N be the level set $f^{-1}(3/2)$. The smooth curve $B := N \cap S$ will be the binding of the open book decomposition we are looking for. It can be proved that the following holds for every regular value a , $0 < a \leq 3/2$:

- the level set $f^{-1}(a)$ is the union along their common boundaries of two surfaces, N_1^a and N_2^a , each one being diffeomorphic to the sub-level surface $S^a := S \cap f^{-1}([0, a])$;
- the sub-level set $M^a := f^{-1}([0, a])$ is divided by S^a into two parts P_1^a and P_2^a which are isomorphic handlebodies (with corners);
- S^a is isotopic to N_i^a through P_i^a , for $i = 1, 2$, by an isotopy fixing its boundary curve $S^a \cap f^{-1}(a)$.

This claim is obvious when a is small and the property is preserved when crossing the critical level 1. In this way the handlebody $H_- := f^{-1}([0, 3/2])$ is divided by $S^{3/2}$ into two diffeomorphic parts $P_i^{3/2}$, $i = 1, 2$, and we have $N = N_1^{3/2} \cup N_2^{3/2}$. We take $S^{3/2}$, $N_1^{3/2}$ and $N_2^{3/2}$ as pages; they are isotopic relative to B through respectively $P_1^{3/2}$ and $P_2^{3/2}$. In $H_+ := f^{-1}([3/2, 3])$, we do the same construction with the function $3 - f$. The open book decomposition is now clear.

We now generalize the notion of open book decomposition.

Definition 3.3. *A generalized open book decomposition of the closed connected 3-manifold M is a pair $\mathcal{B} = (B, \mathcal{P})$ where B is a co-orientable link, the binding, and \mathcal{P} is a codimension-one foliation of $M \setminus B$, whose leaves are called the pages, satisfying the following properties:*

- (1) *The union of each page with the binding is a compact (topological) surface.*
- (2) *In a tubular neighborhood $N(B) \cong D^2 \times B$ of the binding, there are cylindrical coordinates (r, θ, ϕ) , where (r, θ) are polar coordinates in D^2 and ϕ is the projection to B , and $\mathcal{P}|_{N(B)}$ is the foliation defined by $\theta = \text{const}$.*
- (3) *There is at least one page of \mathcal{P} whose intersection with $N(B)$ is a single annulus $\theta = \theta_0$.*

After properties (1) and (2) the space of pages $(M \setminus B)/\mathcal{P}$ is a compact connected 1-manifold, that is: $S^1 := \mathbb{R}/2\pi\mathbb{Z}$ or $I := [-1, +1]$. When the first case holds, \mathcal{B} is a classical open book decomposition.

Definition 3.4. *A generalized open book decomposition \mathcal{B} of M whose space of leaves is the interval I is called a twisted open book decomposition.*

Given a generalized open book \mathcal{B} , consider the projection $p : M \setminus B \rightarrow (M \setminus B)/\mathcal{P}$. In the non-twisted case, each meridian of the binding is mapped by p onto the space of pages S^1 as

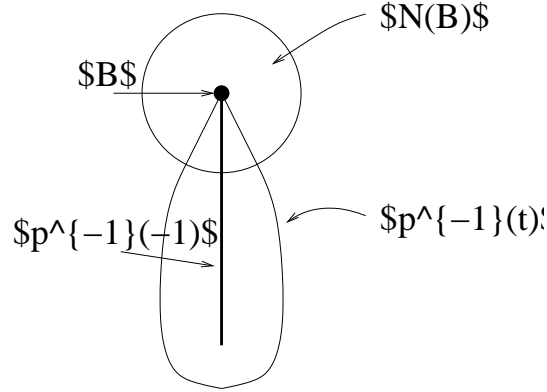


FIGURE 1. Twisted open book.

a regular cover. The property (3) above forces this cover to be of degree one. In the twisted case, property (3) above forces that each meridian meets each regular leaf in two points only.

So, in the twisted case, p is a singular fibration (or *Seifert fibration*) from $M \setminus B$ onto $[-1, +1]$ which has two one-sided exceptional surface fibers $p^{-1}(\pm 1)$, and which is a proper smooth submersion over the open interval. A non-exceptional page is a 2-fold covering of any exceptional page $p^{-1}(\pm 1)$; notice that this covering is trivial over a collar neighborhood of B in the considered exceptional page. The union of an exceptional page with the binding B is a smooth surface with boundary. But the union of a non-exceptional page with B is a closed surface showing (in general) an angle along B .

Notice that, since B is co-orientable, a twisted open book \mathcal{B} gives rise to a smooth foliation $\mathcal{F}_{\mathcal{B}}$ where each component of the binding is replaced with a Reeb component, the pages spiraling around it.

Remark 3.5. We may choose freely the *external holonomy* of this Reeb component, that is, the germ by which the pages are rolled up around the binding, among the germs of diffeomorphisms of $(\mathbb{R}, 0)$ which are the identity on one side and without fixed point on the other side. For the needs of the future homotopy argument at the end of our proof of theorem 1.1 (see section 5), it is essential that this germ ψ be chosen as a product of commutators. Of course, this assumption is known as being easily satisfied: for example, in $Aff(\mathbb{R})$ the unit translation is a commutator; a classical conjugation yields ψ as above (see *e.g.* [14], section 3). Moreover, according to Herman's theorem [9, 10], it is always satisfied; but we want not to use this difficult result.

The *normal bundle* to \mathcal{B} , noted $\nu(\mathcal{B})$, is defined as being the normal bundle $\nu(\mathcal{F}_{\mathcal{B}})$; it is well defined up to isomorphism and is not thought of as a sub-bundle of τM , though it embeds into the tangent bundle τM . Here is the simplest example of a twisted open book.

Example 3.6. Here, $M = S^1 \times S^2$ is thought of as the double of the solid torus $S^1 \times D^2$. The binding is the linear curve $B = (2, 1)$ drawn on the flat separating torus T^2 . One exceptional fiber is the Möbius band \mathbb{M} in the solid torus, and its boundary is $\partial \mathbb{M} = B$. The other exceptional fiber is its mirror copy. Notice that if T^2 is cut along B , one gets an annulus A which doubly covers \mathbb{M} when projecting along the normals to \mathbb{M} . This covering is trivial over a collar

neighborhood of $\partial\mathbb{M}$.

Proposition 3.7. *Let $f : M \rightarrow \mathbb{R}$ be a self-indexing (or ordered) Morse function and S be a one-sided Giroux surface for f . The following middle condition is assumed:*

(MC) *The curve $B := f^{-1}(\frac{3}{2}) \cap S$ is two-sided in its level set $f^{-1}(\frac{3}{2})$.*

Then there is a twisted open book \mathcal{B} whose binding is the curve B and such that $S \setminus B$ is the union of the two exceptional pages.

Proof. Observe first that $S^{3/2} := S \cap f^{-1}([0, \frac{3}{2}])$ is one-sided. Indeed, if a_0 and a_1 are two points in a tubular neighborhood $N(S^{3/2})$ which do not lie on S , they are joined by an arc α in the complement of S as S is one-sided. By using the descending gradient lines of f , we can push α into $N(S^{3/2})$ by an isotopy fixing the end points.

Knowing that the normal bundle to $S^{3/2}$ is twisted, it is possible to construct a piece of twisted open book with binding B inside $N(S^{3/2})$. The gradient of f allows one to extend the open book structure on $f^{-1}([0, \frac{3}{2}])$ so that the complement of B in the level set $f^{-1}(\frac{3}{2})$ is a page. A similar construction in the upper part $f^{-1}([\frac{3}{2}, 3])$ ends to build \mathcal{B} . \square

In general such a one-sided Giroux surface (or a twisted open book) does not exist in M ; the obstruction lies in the existence of a twisted line subbundle of τM (compare theorem 1.2). Right now, we continue the proof of theorem 1.1 starting from the setting which has been stated in corollary 2.6.

We have a decomposition $M = N(\Sigma) \cup H$ where Σ is a closed surface, $N(\Sigma)$ is a I -bundle over Σ , whose projection is $\rho : N(\Sigma) \rightarrow \Sigma$, and H is a handlebody with one or two components. Let d_0 be a small 2-disk in Σ ; set Σ' the closure of $\Sigma \setminus d_0$ and $M' := \rho^{-1}(\Sigma')$. Consider the handlebody $H' := H \cup \rho^{-1}(d_0)$. We have a new decomposition $M = M' \cup H'$. We also recall the line bundle ν which is normal to the Γ_1 -structure ξ under consideration; by assumption it embeds into τM .

Proposition 3.8.

- 1) *When ν is trivial, there exists an open book \mathcal{B} such that Σ' is contained in one page.*
- 2) *When ν is twisted, there exists a twisted open book \mathcal{B} such that Σ' is contained in an exceptional page and the normal bundle $\nu(\mathcal{B})$ is isomorphic to ν .*

The first part is due to E. Giroux [5].

Proof. For beginning with, assume that ν is twisted. The case when ν is trivial admits a similar treatment, with a few modifications which will be specified in the end. According to proposition 2.5 the handlebody H is connected and its genus g is odd; so, the genus of H' is $g + 1$ and even. A basis of compression disks in H' is a family of disjoint compression disks $\mathcal{D} = \{D_0, \dots, D_g\}$ such that cutting H' along \mathcal{D} gives rise to a 3-ball. One passes from one basis to another by a sequence of elementary moves called *slidings*.

Given such a basis, the disk D_k is said to be *orientation-preserving* (resp. *orientation-reversing*) if cutting H' along all the disks other than D_k gives rise to a solid torus (resp. a solid Klein bottle).

The following *sliding property* holds true: if \mathcal{D} is changed to \mathcal{D}' by sliding D_k over D_j and if D_k is orientation-reversing, then the orientation type of D_j in \mathcal{D}' is reversed and the other types remain unchanged.

Lemma 3. *The handlebody H' admits a basis $\mathcal{D} = \{D_0, \dots, D_g\}$ of compression disks verifying:*

- i) D_0, \dots, D_g are disjoint from the small disk $d_0 = \Sigma \cap H'$;
- ii) D_1, \dots, D_g are orientation-preserving;
- iii) H' splits into two connected domains A_0, A_1 whose common boundary is either $d_0 \cup D_0$ (in case H is orientable) or $d_0 \cup D_0 \cup D_g$ (in case H is not orientable);
- iv) For every even (resp. odd) $1 \leq k \leq g-1$, the disk D_k is interior to A_0 (resp. A_1);
- v) In case H is orientable, D_g is interior to A_1 .

Proof of lemma 3.

i), ii) and iii). First let D_1, \dots, D_g be any basis of compression disks for the handlebody H , and choose D_0 a compression disk for H' parallel to d_0 .

In case H is orientable, i) – iii) are immediate.

In case H is non-orientable, we shall change this basis by sliding the disks one over another. There is at least one orientation-reversing D_k , $k \geq 1$. Sliding if necessary D_k over D_0 and D_g , one makes D_0 and D_g orientation-reversing. Then, sliding if necessary D_g over D_1, \dots, D_{g-1} , one makes D_1, \dots, D_{g-1} orientation-preserving. Finally, sliding D_0 over D_g , one makes D_g orientation-preserving. Properties i) – iii) are verified.

iv) and v). One can pass any D_k , $1 \leq k \leq g-1$, as well as D_g in case H is orientable, from A_0 to A_1 , or from A_1 to A_0 , by sliding twice D_0 over D_k . The orientation types are not changed. \square

We now start constructing a one-sided Giroux surface satisfying (MC) and hence, according to proposition 3.7, a twisted open book. The surface Σ' , which is one-sided, is a Giroux surface in M' with respect to some Morse function $f' : M' \rightarrow \mathbb{R}$ having one minimum, $(g+1)$ critical points of index 1 and which is constant on $\partial M'$. Now, we follow Giroux's algorithm for completing Σ' to a closed Giroux surface. First, one has to attach 2-handles to $\partial M'$ in such a way that:

- (vi) each attaching curve intersects $\partial \Sigma'$ in two points exactly.

Thus, each 2-handle will produce simultaneously a 1-handle attached to Σ' , which allows one to extend both f' and the Giroux surface for it (cf. [4]). The previous disks D_0, \dots, D_g are devoted to be cores of these 2-handles, after convenient isotopies: in order that their attaching curves satisfy condition (vi), pairs of intersection points with $\partial \Sigma'$ will be created. The following process is applied in order to control the attachment of the last cell, after the $(g+1)$ surgeries of index 2.

At the first step, a simple arc is drawn on $\partial M' \cap A_1$ from ∂D_1 to $\partial \Sigma'$, otherwise disjoint from the compression disks, and ∂D_1 is pushed by isotopy along this arc in order to create two intersection points with $\partial \Sigma'$. Let (M_1, Σ_1) be the outcome of the handle gluing to (M', Σ') . The boundary of Σ_1 is made of an essential curve c_0^1 and a curve c_1^1 which bounds a disk in ∂M_1 . By an isotopy of M which preserves H' and leaves $D_0, D_2 \dots D_g$ fixed, one makes $c_0^1 = \partial d_0$ and $c_1^1 \subset A_1$.

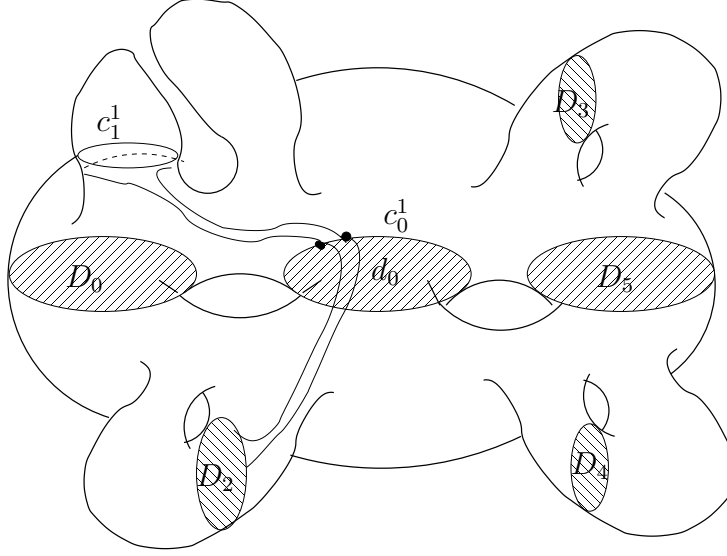


FIGURE 2. The surface ∂M_1 and the isotopy prior to the gluing of D_2 , in case $g = 5$ and H is not orientable.

At the second step, a simple arc is drawn on ∂M_1 from ∂D_2 to c_1^1 , crossing c_0^1 once, otherwise disjoint from the compression disks, and ∂D_2 is pushed by isotopy along this arc in order to create two intersection points with c_0^1 and also to surround c_1^1 ; condition (iv) guarantees that such an arc does exist (compare figure 2).

This surrounding amounts to a handle sliding of D_2 over D_1 . According to condition (ii), this operation does not change the type of the compression disks. And so on, until the gluing of D_g .

After step g , we have (M_g, Σ_g) which is bounded by a torus or a Klein bottle whose complement in M is a handlebody of genus 1 for which D_0 is a compression disk. The boundary of Σ_g is made of an essential curve $c_0^g = \partial d_0$, and a curve c_1^g which is a union of g parallel circles bounding nested disks in $\partial M_g \cap A_1$.

At the last step, a simple arc is drawn on ∂M_g from ∂D_0 to c_1^g , crossing c_0^g , and ∂D_0 is pushed by isotopy along this arc in order to create two intersection points with c_0^g and also to surround c_1^g . This amounts to a handle sliding of D_0 over $D_1 \cup \dots \cup D_g$. By property ii) of lemma 3, when D_0 is orientation reversing, all the other handles become orientation-reversing. After attaching this 2-handle, one gets M_{g+1} whose complement in M is a 3-ball β , and a Giroux surface Σ_{g+1} in M_{g+1} whose boundary is made of $(g+2)$ parallel circles. In order to close the Giroux surface by only one 2-handle, it is necessary to have one circle only. We now explain this last step of Giroux's algorithm.

Let $\gamma_0, \gamma_1, \dots, \gamma_{g+1}$ be the boundary curves of the Giroux surface that we have in $\partial\beta$, the numbering being chosen so that two consecutive circles bound an annulus avoiding the other circles. The regions of their complement in the 2-sphere are colored with two colors alternatively. A trivial 1-handle is attached to M_{g+1} inside β whose core is a simple unknotted arc h_1 in β having one end point in γ_0 and the other in γ_2 ; moreover, one of the attached disks is turned by half a turn in order that the coloring extends along each side of the band which is attached to Σ_{g+1} . Now, there is an obvious 2-handle which kills the previous 1-handle and whose core

satisfies condition (vi) with respect to the previous Giroux surface. After this surgery, we have a Giroux surface in M with a 3-ball removed, whose boundary is made of g parallel circles in a 2-sphere. By repeating this operation we finally get the ideal situation where the boundary of the Giroux surface consists of one curve in the sphere which bounds the *last cell* of M : a 3-cell for closing M containing a 2-cell for closing the Giroux surface.

This construction yields a Giroux surface S equipped with a Morse function f whose critical values are not ordered with respect to their indices in M . We have added *trivial* 1-handles, h_1, h_3, \dots, h_g at a level higher than critical points of index 2.

As in the classical Morse theory it is easy to make the reordering. But, one has to take care of the *middle condition* (MC) in proposition 3.7. This condition could fail only when H' is not orientable. We continue the proof in this case.

The reordering mainly consists of extending the 1-handles h_k 's by the gradient lines of f so that they are thought of as attached to $\partial\Sigma'$. Call $\bar{h}_1, \bar{h}_3, \dots, \bar{h}_g$ these extended 1-handles. One checks that a case when (MC) certainly holds true is when they are compatible with the orientation Or_0 defined near $\partial\Sigma'$. However, by construction, they are compatible with the orientation Or_β of β . As each compression disk of H' , after all the slidings we made, is orientation reversing, each time one crosses one D_k when traversing $\partial\Sigma'$, the sign Or_β/Or_0 changes. Fortunately, the shortest path in $\partial\Sigma'$ joining the feet of \bar{h}_1 crosses exactly two compression disks, as the feet of h_1 are in γ_0 and γ_2 respectively. And similarly for the other \bar{h}_k 's. So, we are done.

Finally, having a Giroux surface and an ordered Morse function at hand, we have a twisted open book structure \mathcal{B} with Σ' in an exceptional page. It remains to check that its normal bundle $\nu(\mathcal{B})$ is isomorphic to ν .

The isomorphism class of a real line bundle is determined by its first Stiefel-Whitney class $w_1(\nu)$, or by its Poincaré dual. By construction, Σ is a Poincaré dual of $w_1(\nu)$ with $\mathbb{Z}/2$ coefficients. On the other hand, the Poincaré dual of $w_1(\nu(\mathcal{B}))$ is the Giroux surface S that we have built. But, d_0 and $S \setminus \text{int}(\Sigma')$ are homologous in $(H', \partial H')$ since they share a common boundary in the aspherical manifold H' . Therefore, S and Σ are homologous and the considered bundles are isomorphic.

The case when ν is trivial is very similar. Here, Σ is two-sided and the handlebody H has two connected components, $H = H_1 \sqcup H_2$, each one having the same genus g . The handlebody H' is made of the union of H and the 1-handle $\rho^{-1}(d_0)$; it has genus $2g$. Since the two handlebodies have diffeomorphic boundaries, if one is non-orientable, the other is neither.

There are compression disks $D_1, D_3, \dots, D_{2g-1}$ in H_1 and compression disks D_2, D_4, \dots, D_{2g} in H_2 . All together they are compression disks which cut H' into one ball exactly. So, they will become the core of 2-handles attached to M' after some suitable isotopy of their attaching curves. The boundary of D_1 is moved so that it intersects $\partial\Sigma'$ in two points. Next, we move the boundary of D_2 and so on. It is easy to check that the configuration of nested disks, similar to the previous case, can be realized at each step. \square

Corollary 3.9. *Theorem 1.2 holds true.*

Proof. Start with the trivial or twisted trivial Γ_1 -structure ξ_0 according to whether its normal bundle is trivial or not (compare definition 2.2). Apply the Morsification process (section 2)

until the decomposition

$$M = N(\Sigma) \cup H$$

of corollary 2.6. Thus, proposition 3.8 yields the conclusion. \square

3.10. Modification by suspension. We end this section by explaining how the foliation $\mathcal{F}_{\mathcal{B}}$ associated above to any twisted open book \mathcal{B} , can be modified by suspension in a similar manner as we did in section 1 for open book foliations.

Let K be a compact subsurface in the exceptional page $p^{-1}(-1)$. Consider a tubular neighborhood $N(K)$ of K in M , indeed a $[-\varepsilon, \varepsilon]$ -bundle over K , which is compatible with $\mathcal{F}_{\mathcal{B}}$ in the following sense: the trace of $\mathcal{F}_{\mathcal{B}}$ on $N(K)$ is the foliation suspension of the restricted representation

$$\varphi_0 := w_1(\nu) : \pi_1(K) \rightarrow \mathbb{Z}/2 = \text{Aut}([-\varepsilon, \varepsilon])$$

(compare definition 2.2). Let $\varphi : \pi_1(K) \rightarrow \text{Diff}([-\varepsilon, \varepsilon])$ be a representation such that:

1) for each $\alpha \in \pi_1(K)$ and $x \in [-\varepsilon, \varepsilon]$ close to the end points, we have:

$$\varphi(\alpha)(x) = \varphi_0(\alpha)(x);$$

2) if α is peripheral, $\varphi(\alpha) = \varphi_0(\alpha)$.

These two conditions allow us to remove $\mathcal{F}_{\mathcal{B}}$ from the interior of $N(K)$ and replace it with the suspension of φ . The modified foliation of M is denoted by $\mathcal{F}_{\mathcal{B}, \varphi}$.

Definition 3.11. Any foliation of this form is said to be carried by the twisted open book \mathcal{B} .

After this definition, the statement of theorem 1.1 is meaningful.

4. HOMOTOPY TO THE PLATEAU FORM

This section is a more step toward proving the regularization theorem 1.1. We introduce the following definition.

Definition 4.1. Given a Γ_1 -structure ξ on a space G , by an upper (resp. lower) completion of ξ one means a foliation \mathcal{F} of $G \times (-\varepsilon, 1]$ (resp. $G \times [-1, \varepsilon)$), for some positive ε , which is transverse to every fiber $\{x\} \times (-\varepsilon, 1]$ (resp. $\{x\} \times [-1, \varepsilon)$), whose germ along $G \times \{0\}$ is ξ , and such that $G \times \{t\}$ is a leaf of \mathcal{F} for every t close enough to $+1$ (resp. -1).

Proposition 4.2. Every co-orientable Γ_1^r -structure ξ on a simplicial complex G of dimension 1, $r \geq 1$, admits an upper (resp. lower) completion of class C^r .

Proof. Let us show the lower completion. After a fine subdivision of the edges, one reduces oneself to the case where, over each edge α of G , the germ of foliation ξ is given by the level sets of a real function $f(x, t)$, $(x, t) \in \alpha \times]-\varepsilon, +\varepsilon[$; this function is smooth in x , is C^r in t , and satisfies $\frac{\partial f}{\partial t} > 0$ everywhere. Moreover, we may assume that the completion is already given over the 0-skeleton of G . At this point, we argue as in proposition 2.1: keeping the germ of f fixed along $\alpha \times \{0\} \cup G^{[0]} \times [-1, 0]$ one can arrange that f is C^∞ near $t = -\varepsilon$. Finally, the dimension of $\alpha \times [-1, 0]$ being 2, we are reduce to build a smooth line field which fulfills the statement; the flow lines of this line field are the leaves of the wanted foliation. This can be done by partition of unity. \square

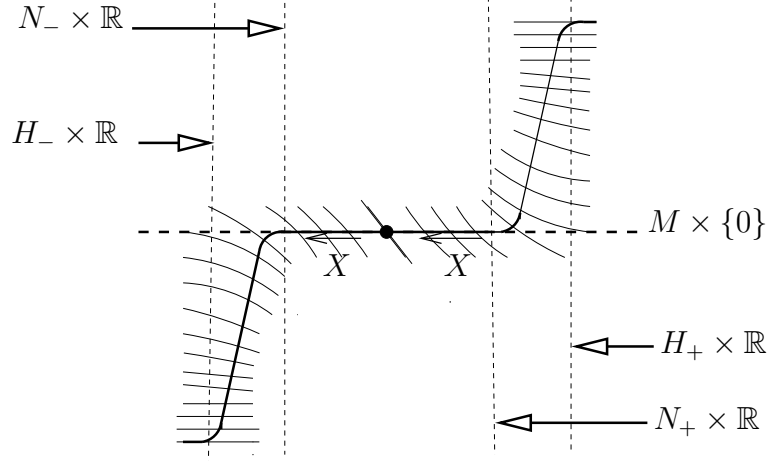


FIGURE 3. Homotopy to the plateau form tracked by some orbit of X .

Now we come back to the proof of theorem 1.1. Recall the Morse Γ_1^r -structure ξ obtained in proposition 2.4, with its normal bundle ν , its pseudo-gradient X , and the associated decomposition $M = N(\Sigma) \cup H$ (proposition 2.5, corollary 2.6).

Definition 4.3. A Γ_1 -structure ξ_{plat} on M with normal bundle ν is said to be in plateau form with respect to this decomposition, if:

- ξ_{plat} is trivial over H ;
- ξ_{plat} is regular over the interior of $N(\Sigma)$ and, in this domain, it is transverse to the fibers of $N(\Sigma) \rightarrow \Sigma$.

In other words, ξ_{plat} is a suspension foliation in $N(\Sigma)$ and trivial in H .

Proposition 4.4. The Γ_1^r -structure ξ is homotopic to one, noted ξ_{plat} , in plateau form with respect to the decomposition $M = N(\Sigma) \cup H$.

This move is due to T. Tsuboi in [19], where it is given as an exercise.

Proof. The homotopy of Γ_1 -structures will actually be a homotopy of the zero section in a foliated domain of the total space $E(\nu)$. We give the proof when ν is trivial only, the other cases being similar. Then, $E(\nu) = M \times \mathbb{R}$, and X is a genuine vector field, and the graph G formed by the separatrices of X splits into $G_+ \sqcup G_-$, where G_+ (resp. G_-) is a repeller (resp. an attractor) of X . By proposition 4.2, ξ admits an upper (resp. a lower) completion over G_+ (resp. G_-), and thus also over an open neighborhood N_+ (resp. N_-) of G_+ (resp. G_-) in M . Recall that in section 2.1 H was defined as an arbitrarily small handlebody neighborhood of G whose boundary is transverse to X . Thus, we can arrange that the connected component H_+ (resp. H_-) of H containing G_+ (resp. G_-) is contained in N_+ (resp. N_-). So we have a foliation \mathcal{F} defined on a neighborhood of

$$(M \times \{0\}) \cup (H_- \times [-1, 0]) \cup (H_+ \times [0, 1])$$

in $M \times \mathbb{R}$ which is transverse to X on $(M \setminus \text{int}(H_- \cup H_+)) \times \{0\}$ and tangent to $H_{\pm} \times \{t\}$ for every t close to ± 1 .

Recall (section 2.1) that there is a diffeomorphism

$$F : M \setminus \text{Int}(H'_- \cup H'_+) \rightarrow \Sigma \times [-1, +1]$$

which maps orbit segments of X onto fibers.

For a small $\varepsilon > 0$, choose a function $\psi : \mathbb{R} \rightarrow [-1, +1]$ which is smooth, odd, and such that:

- $\psi(t) = 0$ for $0 \leq t \leq 1 - 3\varepsilon$ and $\psi(1 - 2\varepsilon) = \varepsilon$;
- ψ is affine on the interval $[1 - 2\varepsilon, 1 - \varepsilon]$;
- $\psi(1 - \varepsilon) = 1 - \varepsilon$ and $\psi(t) = 1$ for $t \geq 1$;
- $\psi' > 0$ on the interval $]1 - 3\varepsilon, 1[$.

Let $s : M \rightarrow M \times \mathbb{R}$ be the graph of the function whose value is ± 1 on H_{\pm} and $\psi(t)$ at the point $F^{-1}(x, t)$ for $(x, t) \in \Sigma \times [-1, +1]$. When ε is small enough, it is easily checked that, for every $x \in \Sigma$, the path $t \mapsto s \circ F^{-1}(x, t)$ is transverse to \mathcal{F} except at its end points. Then, $\xi_1 := s^*\mathcal{F}$ is homotopic to ξ and obviously fulfills the conditions required in proposition 4.4. Indeed, at each point, $\langle s_*X, \xi \rangle$ is a non-negative linear combination of $\langle X, \xi \rangle$ and $\langle -\frac{\partial}{\partial t}, \xi \rangle$, hence, it is non-vanishing. (cf. figure 3) \square

5. HOMOTOPY OF Γ_1 -STRUCTURES

Now we complete the proof of theorem 1.1.

On the one hand, to the given Γ_1 -structure ξ was associated a decomposition $M = H \cup N(\Sigma)$, and according to proposition 4.4, ξ was homotoped to ξ_{plat} in plateau form with respect to this decomposition. So, ξ_{plat} is trivial on the handlebody H , while on $N(\Sigma)$, it is the suspension foliation of some representation $\varphi : \pi_1(\Sigma) \rightarrow \text{Diff}([-1, +1])$. The gluing with the trivial Γ_1 -structure on H implies that, for each $\alpha \in \pi_1(\Sigma)$ and $x \in [-1, +1]$ near the end points, we have:

$$\varphi(\alpha)(x) = \varphi_0(\alpha)(x),$$

where φ_0 is obtained by restricting the first Stiefel-Whitney class of ν (compare condition 1) in 3.10). Recall that d_0 is a small disk in Σ , and $\Sigma' = \Sigma \setminus \text{int}(d_0)$.

On the other hand, according to proposition 3.8, Σ' is contained in a page of some open book \mathcal{B} ; when ν is twisted, \mathcal{B} is twisted and Σ' lies in an exceptional page. Applying 3.10, we modify $\mathcal{F}_{\mathcal{B}}$ by the suspension of the above-mentioned representation φ and get a foliation $\mathcal{F}_{\mathcal{B}, \varphi}$. By construction, this foliation coincides with ξ_{plat} in $N(\Sigma') \cong \rho^{-1}(\Sigma')$, where ρ is the projection $N(\Sigma) \rightarrow \Sigma$.

Recall that H' is the handlebody which is the union of H and $\rho^{-1}(d_0)$. To establish theorem 1.1, it remains to prove that the plateau Γ -structure ξ_{plat} and the Γ -structure $\mathcal{F}_{\mathcal{B}, \varphi}$ carried by \mathcal{B} are homotopic as Γ_1 -structures on H' relative to $\partial H'$. Notice that this statement is in fact independent of the representation φ since the modification of $\mathcal{F}_{\mathcal{B}}$ into $\mathcal{F}_{\mathcal{B}, \varphi}$ did not modify the foliation in H' .

The homotopy will be done in two steps, starting from $\mathcal{F}_{\mathcal{B}, \varphi}$.

First step. First, one flattens its Reeb components. Set $N_1(B)$ the tubular neighborhood of B which is the union of the Reeb components of $\mathcal{F}_{\mathcal{B}, \varphi}$. As we are speaking here of non-thickened Reeb components we have $N_1(B) \subset N(B)$; the collar between both tubes is foliated by tori.

Lemma 4. *There exists a homotopy, relative to $M \setminus \text{int}(N_1(B))$, from $\mathcal{F}_{\mathcal{B},\varphi}$ to a Γ_1 -structure ξ_φ on M which is trivial on $N_1(B)$.*

Proof. Let R be any connected component of $N_1(B)$. Since the Reeb components of $\mathcal{F}_{\mathcal{B},\varphi}$ are *thick*, the holonomy of ∂R is trivial outside. This holonomy is generated by the germ at 0 of some self-diffeomorphism λ of the real line, whose support is contained in $[0, +\infty)$. Let \mathcal{F}_λ be the suspension of λ . It is a foliation on the annulus $S^1 \times \mathbb{R}$, whose closed leaves are $S^1 \times \{t\}$, $t \leq 0$.

Then, on some small neighborhood $N(R)$ of R , the foliation $\mathcal{F}_{\mathcal{B},\varphi}$ is the pullback of \mathcal{F}_λ by some smooth map $F : N(R) \rightarrow S^1 \times \mathbb{R}$. More precisely, in cylindrical coordinates (r, θ, ϕ) of $R \cong D^2 \times S^1$ where ϕ is the normal projection onto S^1 and D^2 is of radius 1, we take $F(r, \theta, \phi) := (\phi, \epsilon(1 - r^2))$ with $\epsilon > 0$ small.

Write $F(x) = (s(x), f(x))$ and define $G(x) := (s(x), g(f(x)))$ where g is any smooth function on the real line such that $g(t) = 0$ for every $t \geq 0$ and that $g'(t) > 0$ for every $t < 0$. Then, $\xi_\varphi := G^*\mathcal{F}_\lambda$ obviously works. The homotopy from $\mathcal{F}_{\mathcal{B},\varphi}$ to ξ_φ is clear. \square

In the second step ξ_φ will be homotoped to ξ_{plat} relative to $\partial H'$. This second step will be different depending on whether ν is trivial or not.

Second step, co-orientable case. Here the bundle ν is assumed to be trivial and \mathcal{B} is a genuine open book decomposition.

After the following lemma we shall be done with the homotopy problem.

Lemma 5. *There exists a homotopy from ξ_φ to ξ_{plat} relative to $N(\Sigma')$.*

Proof. Recall the decomposition $M = N(\Sigma') \cup H'$. We have to prove that the restrictions of ξ_φ and ξ_{plat} to H' are homotopic relative to $\partial H'$. Consider the standard closed 2-disk $D = D^2$ endowed with the Γ_1 -structure ξ_D which is shown on figure 4.

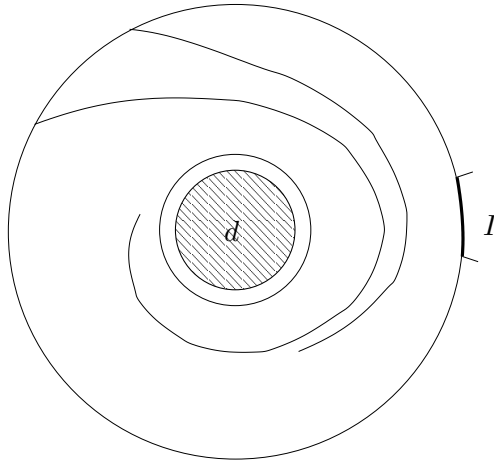


FIGURE 4. The Γ_1 -structure ξ_D on the 2-disk.

It is trivial on the small disk d and regular on the annulus $D \setminus \text{int}(d)$. In the regular part, the leaves are circles near ∂d and the other leaves are spiraling, crossing ∂D transversely. The

restriction of ξ_φ to H' has the form $f^*\xi_D$ for some map $f : H' \rightarrow D$. Namely, on $N(B) \cong D \times S^1$ we take for f the canonical projection onto D ; and we extend it continuously to H' by sending every leaf of $\xi_\varphi|_{H'}$, which is a subset of a page of \mathcal{B} , to a point of ∂D . In particular, f sends the 2-dimensional cylinder $\rho^{-1}(\partial d_0)$ to an interval I embedded in ∂D .

In a similar way, $\xi_{\text{plat}}|_{H'}$ is the pullback of ξ_D by the map $g : H' \rightarrow I \subset D$ which equals f on $\partial H'$ and sends H to ∂I (recall that H has two connected components in the co-orientable case). Each regular leaf of ξ_{plat} is sent to the same point of I as its trace on $\rho^{-1}(d_0)$. Since D retracts by deformation onto I , consequently ξ_φ is homotopic to ξ_{plat} on H' relative to $\partial H'$. \square

This finishes the proof of theorem 1.1 in the co-orientable case.

Second step, twisted case. In the twisted case, the homotopy argument will be a little more sophisticated, because the classifying space of the Seifert fibrations over the interval is infinite-dimensional, as the classifying space of the line bundles is.

In the compact unit 3-ball D^3 , let σ be the orientation-preserving involution of D^3 defined by $\sigma(x, y, t) = (x, -y, -t)$.

Lemma 6. *There is a σ -invariant Γ_1 -structure ξ_σ on D^3 , whose restriction to the disk $(\partial D^3) \cap \{t \leq 0\}$ is conjugate to the Γ_1 -structure ξ_D represented on figure 4.*

Proof. It is more convenient to regard D^3 as the solid cylinder, namely, the product $D^2 \times [-1, +1]$. In this model one has $\sigma(z, t) = (\bar{z}, -t)$, where $z = x + iy$.

The trace of ξ_σ on $\partial(D^2 \times [-1, +1])$ will be trivial over $D^2 \times \{\pm 1\}$ and regular on $\partial D^2 \times [-1, +1]$, where it will indeed be the suspension of some diffeomorphism ζ of the interval $[-1, +1]$. We first build this diffeomorphism.

Recall from remark 3.5 that the rolling-up germ ψ is chosen as a product of commutators. Let $0 < \epsilon < 1/2$. One easily makes a diffeomorphism γ of the interval $[-1, +1]$ such that:

- a) One has $\gamma(t) \geq t$ with equality if and only if $t \geq 1 - 2\epsilon$ or $t \leq -1 + \epsilon$;
- b) The germ of γ at $-1 + \epsilon$ is conjugate to ψ ;
- c) γ is a product of commutators: $\gamma = [\alpha_1, \beta_1] \dots [\alpha_g, \beta_g]$. Here each α_j , each β_j is a diffeomorphism of the interval $[-1, +1]$ with support in $(-1, +1)$; and $[\alpha, \beta]$ denotes $\alpha\beta\alpha^{-1}\beta^{-1}$.

In order to construct such a γ one starts from a factorization of ψ into g commutators and extends each entry as a diffeomorphism of $[-1, +1]$.

Then, set $\zeta := [\tau, \gamma^{-1}]$ where $\tau(t) := -t$. One has:

- i) The germ of ζ at $-1 + \epsilon$ is conjugate to ψ ;
- ii) $\tau\zeta\tau = \zeta^{-1}$. In other words, the suspension of ζ is σ -invariant in $\partial D^2 \times [-1, +1]$;
- iii) $\zeta(t) \geq t$. Indeed, for every $u \in [-1, +1]$ one has $\gamma^{-1}(u) \leq u \leq \tau\gamma^{-1}\tau(u)$. Applying this at $u = \gamma(t)$ one has $t \leq \gamma(t) \leq [\tau, \gamma^{-1}](t)$.
- iv) One has $\zeta(t) = t$ if and only if $t = \gamma(t)$ and $\tau(t) = \gamma(\tau(t))$; that is, $t \leq -1 + \epsilon$ or $t \geq 1 - \epsilon$.

One will define ξ_σ as the foliation given in $D^2 \times [-1, +1]$ by the height function $h(z, t) := t$, modified as follows. Create in the interior of $D^2 \times [-1, +1]$ a number g of pairs of singularities,

the j -th pair ($j = 1, \dots, g$) consisting of two singularities s_1^j, s_2^j of respective indices 1, 2, in cancellation position. Let f be the resulting Morse function. One chooses its singular values so that $f(s_1^j) < -1 + \epsilon < 1 - \epsilon < f(s_2^j)$ ($j = 1, \dots, g$). So, the intermediate level sets have got some genus: for $u \in [-1 + \epsilon, 1 - \epsilon]$, the level set $f^{-1}(u)$ is a compact surface of genus g bounded by one circle. Its fundamental group F_{2g} being non-abelian free on $2g$ generators, the α_j 's and β_j 's define a representation $\lambda : F_{2g} \rightarrow \text{Diff}([-1, +1])$. Next, in $f^{-1}[-1 + \epsilon, 1 - \epsilon]$, one changes the level surfaces of f to the suspension of the representation λ . The resulting Γ_1 -structure on $D^2 \times [-1, +1]$ induces the suspension of γ on $(\partial D^2) \times [-1, +1]$. One pushes this structure by some convenient isotopy to make it coincide with the height function in the half cylinder $\{y \geq 0\} \times [-1, +1]$. Next, in the half cylinder $\{y \leq 0\} \times [-1, +1]$, one performs the modification which is σ -symmetric to the preceding one. Finally, we obtain a σ -invariant Γ_1 -structure ξ_σ in the solid cylinder.

Obviously ξ_σ is trivial over $D^2 \times \pm 1$, while its trace on $(\partial D^2) \times [-1, +1]$ is the suspension of the diffeomorphism $[\tau, \gamma^{-1}]$. Thus, thanks to property i) above, the trace of ξ_σ on the disk $(D^2 \times -1) \cup (\partial D^2 \times [-1, 0])$ is conjugate to ξ_D . □

The involution σ is suspended to get a D^3 -bundle over $\mathbb{R}P^\infty$. Let E be its total space. By lemma 6, there exists a Γ_1 -structure ξ_E on E whose restriction to each fiber is ξ_σ . Its normal bundle is the unique twisted bundle of rank one over E .

Lemma 7. *There are two continuous maps $f, g : H' \rightarrow E$, equal on $\partial H'$, such that $\xi_\varphi|_{H'} = f^*\xi_E$ and $\xi_{\text{plat}}|_{H'} = g^*\xi_E$.*

Proof. Consider the circle $S^1 := \{t = 0\}$ in S^2 in the coordinates considered above. As it is σ -invariant, it defines a circle subbundle $E' \subset E$ over $\mathbb{R}P^\infty$. Every circle fiber is transverse to ξ_E . Actually $\xi_E|_{E'}$ coincides with the (infinite-dimensional) horizontal foliation $\mathcal{F}_{E'}$ on E' which is the suspension over $\mathbb{R}P^\infty$ of the orientation-reversing involution of the circle, $\sigma|_{S^1}$. The holonomy covering space of each leaf is S^∞ , thus contractible. So, $(E', \mathcal{F}_{E'})$ is the Haefliger classifying space for the groupoid generated by this involution ([8]). In other words, $(E', \mathcal{F}_{E'})$ is the classifying space for foliations of codimension one which are *wandering*, that is, every leaf meets every transverse interval in a finite set. Any such foliation \mathcal{F} on any manifold V is the pullback of $\mathcal{F}_{E'}$ by some *classifying map* $V \rightarrow E'$. One also has the relative version: if $X \subset V$ is a submanifold transverse to \mathcal{F} then every classifying map for $\mathcal{F}|_X$ extends to some classifying map for \mathcal{F} .

Just as in the orientable case above, $\xi_\varphi|_{N(B)}$ is the pullback of ξ_D through the canonical projection $N(B) \cong D^2 \times S^1 \rightarrow D^2$. On the other hand, in the complement $H' \setminus \text{int}(N(B))$ the Γ_1 -structure ξ_φ is a wandering foliation. Embed D^2 into E as one half of the boundary of some 3-ball fibre, so that ξ_D is the restriction of ξ_E to D^2 (lemma 6). Thanks to the relative classifying property of $(E', \mathcal{F}_{E'})$, this projection extends to some map $f : H' \rightarrow E$ such that $\xi_\varphi|_{H'} = f^*\xi_E$. Recall that ξ_φ and ξ_{plat} coincide on $\partial H'$. Thus, again by the relative classifying property, $\xi_{\text{plat}}|_{H'}$ is the pullback of $\mathcal{F}_{E'}$ by some map $g : M \rightarrow E'$ equal to f on $\partial H'$. If one likes better, rather than invoking Haefliger's classifying property [8], one can in this case easily build f and g by hands.

□

Finally, H' being a handlebody, and $\pi_2(E)$, $\pi_3(E)$ being both trivial, necessarily f and g are homotopic rel. $\partial H'$. So ξ_φ and ξ_{plat} are homotopic in H' rel. $\partial H'$. This completes the homotopy argument in the twisted case, and the proof of theorem 1.1.

REFERENCES

- [1] J.W. Alexander, *Note on Riemann spaces*, Bull. Amer. Math. Soc. 26 (1920), 370-372.
- [2] J.W. Alexander, *A lemma on systems of knotted curves*, Proc.Nat. Acad. Sci. U.S.A. 9 (1923), 93-95.
- [3] H. Geiges, *An Introduction to Contact Topology*, .Cambridge Univ.Press, 2008.
- [4] E. Giroux, *Convexit  en topologie de contact*, Comment. Math. Helv. 66 (1991), 637-677.
- [5] E. Giroux, *private communication*.
- [6] E. Giroux, N. Goodman, *On the stable equivalence of open books in three-manifolds*, Geometry & Topology 10 (2006), 97-114.
- [7] A. Haefliger, *Homotopy and integrability*, 133-175 in: Manifolds-Amsterdam 1970, L.N.M. 197, Springer, 1971.
- [8] A. Haefliger, *Groupo ides d'holonomie et classifiants*, Ast risque 116 (1984), 70-97.
- [9] M.R. Herman, *Simplicit  du groupe des diff omorphismes de classe C^∞ , isotopes   l'identit , du tore de dimension n* , C. R. Acad. Sci. Paris, S r. A-B, 273 (1971), A232-A234.
- [10] M.R. Herman, *Sur le groupe des diff omorphismes du tore*, Ann. Inst. Fourier (Grenoble) 3 (1973), 75-86.
- [11] F. Laudenbach, G. Meigniez, *Regularization of Γ_1 -structures in dimension 3*, arXiv: 0906.1748.
- [12] J. N. Mather, *On Haefliger's classifying space I*, Bull. Amer. Math. Soc. 77 (1971), 1111-1115.
- [13] J. N. Mather, *Integrability in codimension 1*, Comment. Math. Helv. 48 (1973), 195-233.
- [14] G. Meigniez, *A compactly generated pseudogroup which is not realizable*, J. Math. Soc. Japan 62, no. 4 (2010), 1205-1218.
- [15] G. Meigniez, *Regularization and minimization of Γ_1 -structures*, arXiv: math/GT/0904.2912.
- [16] F. Sergeraert, *Feuilletages et diff omorphismes infiniment tangents   l'identit *, Inventiones Math. 39 (1977), 253-275.
- [17] W. Thurston, *Foliations and groups of diffeomorphisms*, Bull. Amer. Math. Soc. 80 vol. 2 (1974), 304-307.
- [18] W. Thurston, *Existence of codimension-one foliations*, Annals of Math. 104 (1976), 249-268.
- [19] T. Tsuboi, *Classifying spaces for groupoid structures*, Conference *Foliations* (Rio de Janeiro, 2001). Online at <http://www.foliations.org/surveys>.
- [20] F. W. Wilson, *On the minimal sets of non-singular vector fields*, Annals of Math. 84 (1966), 529-536.
- [21] H. Winkelnkemper, *Manifolds as open book*, Bull. Amer. Math. Soc. 79 (1973), 45-51.

UNIVERSIT  DE NANTES, UMR 6629 DU CNRS, 44322 NANTES, FRANCE

E-mail address: francois.laudenbach@univ-nantes.fr

UNIVERSIT  DE BRETAGNE-SUD, L.M.A.M., BP 573, F-56017 VANNES, FRANCE

E-mail address: Gael.Meigniez@univ-ubs.fr